

# Measurement uncertainty: the problem of characterising optimal error bounds

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We consider the task of characterizing optimal protocols for approximating incompatible observables via jointly measurable observables. This amounts to jointly minimizing the approximation errors (suitably quantified), subject to the compatibility constraint. We review two distinct ways of conceptualizing the joint measurement problem and elucidate their connection. As a case study we consider the approximation of two-valued qubit observables and scrutinize two recent approaches that are based on different ways of quantifying errors, each giving rise to a form of tradeoff relation for the error measures used. For the first of these approaches we exhibit a formulation of the tradeoff in operational terms as a *measurement uncertainty relation*. Furthermore we find a disparity between the respective optimal approximators singled out by the two approaches, which underlines the operational shortcomings of the second type of error measures.

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## I. INTRODUCTION

Incompatibility is a ubiquitous feature of quantum theory, with even the simplest quantum systems possessing observables that cannot be measured together. One way of mitigating the impossibility of realizing accurate joint measurements of two incompatible quantities is to approximate them by two observables that are jointly measurable, as shown in Fig. 1.

The degree to which an approximating observable may be considered to be an accurate representation of the initial observable, referred to here as a *target observable*, is given by a measure of *error*. The resulting problem of formulating *measurement uncertainty relations* for incompatible quantum observables has come into focus only rather recently, and the associated question of how to appropriately quantify measurement errors is the subject of an ongoing controversy [1].

When jointly approximating two incompatible observables  $A$  and  $B$  by measuring compatible observables  $C$  and  $D$ , respectively, the associated errors are quantified by some (as yet unspecified) measure  $\delta$ . These error pairs  $(\delta(A, C), \delta(B, D))$  fill an area, called the *admissible region*, as the approximators  $C, D$  are varied across all possibilities, subject to the compatibility constraint. Within this region, the area of greatest interest is the minimum error bound, describing the smallest possible errors that we may obtain whilst still demanding joint measurability.

The problem of determining the optimal boundary of the admissible error region for approximate joint measurements was formulated and discussed qualitatively in the case of dichotomic (two-valued) qubit observables in [2]; a linear tight approximation was proven for a state-independent, metric measure of error, defined as a distance ( $D$ ) between observables (see also [3]). The exact boundary curve for this case was presented recently by Yu

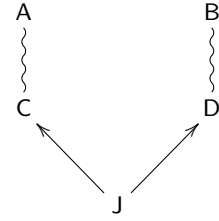


FIG. 1. Two incompatible observables  $A$  and  $B$  are approximated via jointly measurable observables  $C$  and  $D$ , with joint observable  $J$ . The error measures act as figures of merit of how well  $C$  ( $D$ ) approximates  $A$  ( $B$ ).

and Oh [4]. This bound, whilst shown to be attainable, was originally described parametrically and the operational meaning of the terms given was not made clear.

Another approach aiming at establishing tight error bounds was given by Branciard [5]; it is formulated in terms of a measure ( $\varepsilon$ ) of *measurement noise* proposed by Ozawa [6] and the bound obtained is shown to be an attainable, state-dependent, lower bound for any two incompatible observables in any dimension of Hilbert space considered. However, the adequacy of the noise measure as a quantification of measurement error in this approach has been called into question [1].

Here we present a case study to provide a comparison of the two competing proposals of error measures—the metric error on the one hand and the measurement noise on the other. We show how these conceptually different quantities lead to divergent assessments of what may be regarded as an optimal joint approximate measurement.

We begin by examining the method of producing jointly measurable approximating observables used by Branciard, which follows the work of Hall [7] and is expressed in terms of a comparison of selfadjoint operators that are taken to represent the target observable and approximating measurements, respectively. By contrast, the approach followed in [2, 3] formalizes the notion

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of joint measurement in terms of joint probabilities and quantifies approximation errors by means of metrics on the space of observables, represented as positive operator valued measures (POVMs). To enable a comparison of the different methodologies, we provide a translation between the two conceptual schemes applied.

Yu and Oh gave a purely geometric description of the boundary curve of the admissible region for metric errors; we characterize this curve in physical terms, showing how the errors are constrained by an interplay between the incompatibility of the target observables and the degrees of unsharpness of the approximating observables; along the way we uncover and fix a shortcoming in the original argument of [4].

We also describe the sets of optimizing approximators  $C, D$ , both in the case of the metric error  $D$  and the measurement noise  $\varepsilon$ , and find that these sets are quite different, albeit not without overlap in some cases.

## II. PRELIMINARIES

In what follows we consider qubit systems described by the Hilbert space  $\mathcal{H} = \mathbb{C}^2$ . We denote *vector states* by unit vectors  $\psi, \xi \in \mathcal{H}$  and (generally) mixed states by density operators (positive operators of unit trace)  $\rho$ . Vector states correspond to the pure states in the convex set of density operators, where they are represented as rank-1 projection operators  $P_\psi = |\psi\rangle\langle\psi|$ .

Among the set of self-adjoint linear operators acting on  $\mathcal{H}$ , a distinguished subset is given by the set of *effects*; these are the operators in the interval  $0 \leq E \leq I$ , where  $0, I$  are the null and identity operator, respectively, and  $\leq$  denotes the usual operator ordering ( $A \leq B$  if  $\langle\psi|A\psi\rangle \leq \langle\psi|B\psi\rangle$  for all  $\psi \in \mathcal{H}$ ).

In this paper we consider only discrete observables, associated with positive operator-valued measures (POVMs; also called effect-valued measures): a POVM  $C$  in  $\mathcal{H}$  is defined as a map  $c_i \mapsto C(c_i) = C_i$  for a discrete set  $c_i \in \{c_1, \dots, c_N\}$  where the  $C_i$  are effects such that  $\sum_i C_i = I$ . Throughout, we denote POVMs by sans serif letters. Sharp observables are described by projection-valued (or spectral) measures (PVMs): a PVM is a POVM  $P$  whose effects  $P_i = P(p_i)$  are projection operators ( $P_i^2 = P_i$ ) for all  $p_i \in \{p_1, \dots, p_M\}$ . All other POVMs will be called unsharp observables. For a POVM  $C$  with effects  $\{C_1, \dots, C_N\}$  we define the  $k^{\text{th}}$  moment operator  $C[k]$ ,  $k \in \mathbb{N}$ , via

$$C[k] = \sum_i c_i^k C_i. \quad (1)$$

Note that it will usually be the case that  $C[k] \neq C[1]^k$  (unless  $C$  is sharp).

We shall make use of the Bloch representation of operators acting on  $\mathbb{C}^2$ : A selfadjoint operator  $A$  can be expressed in the form

$$A = \frac{1}{2}(\alpha I + \mathbf{a} \cdot \boldsymbol{\sigma}), \quad (2)$$

where  $\alpha \in \mathbb{R}$ ,  $\mathbf{a} \in \mathbb{R}^3$ , and  $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$  is the vector composed of the standard Pauli matrices. The eigenvalues of  $A$  are equal to  $\frac{1}{2}(\alpha \pm \|\mathbf{a}\|)$ , and so  $A$  is an effect if and only if  $\|\mathbf{a}\| \leq \alpha \leq 2 - \|\mathbf{a}\|$  (equivalently,  $\|\mathbf{a}\| \leq \min\{\alpha, 2 - \alpha\}$ ). Similarly, a density operator  $\rho = \frac{1}{2}(I + \mathbf{r} \cdot \boldsymbol{\sigma})$  is characterized by  $0 \leq \|\mathbf{r}\| \leq 1$ .

We will focus predominantly on dichotomic ( $\pm 1$ -valued) POVMs, which possess two effects. A dichotomic qubit POVM  $C$  has effects  $C(\pm) = C_\pm$  of the form

$$C_\pm = \frac{1}{2}((1 \pm \gamma)I \pm \mathbf{c} \cdot \boldsymbol{\sigma}), \quad (3)$$

where  $|\gamma| + \|\mathbf{c}\| \leq 1$ . We say that  $C$  is *symmetric* (or *unbiased*) if  $\gamma = 0$  [3].

Two observables  $E$  and  $F$  (defined on finite sets  $\Omega_1$  and  $\Omega_2$ , respectively) are said to be *jointly measurable* if there exists an observable  $J$  (defined on  $\Omega_1 \times \Omega_2$ ) such that

$$\begin{aligned} E(k) &= \sum_{\ell \in \Omega_2} J(k, \ell), \\ F(\ell) &= \sum_{k \in \Omega_1} J(k, \ell). \end{aligned} \quad (4)$$

for any effects  $E(k), F(\ell)$ . Such a  $J$  is called a *joint observable* of  $E$  and  $F$ , and equivalently  $E$  and  $F$  are *marginal observables* (or *margins*) of  $J$ . A sufficient condition for joint measurability of  $E$  and  $F$  is commutativity of their effects, i.e.,  $[E(k), F(\ell)] = 0$  for all effects  $E(k)$  of  $E$  and  $F(\ell)$  of  $F$ , but this condition is not necessary unless at least  $E$  or  $F$  is sharp. If  $E$  and  $F$  are symmetric dichotomic qubit observables, then joint measurability is equivalent [8] to the Bloch vectors  $\mathbf{e}$  and  $\mathbf{f}$  describing their effects satisfying

$$\|\mathbf{e} + \mathbf{f}\| + \|\mathbf{e} - \mathbf{f}\| \leq 2. \quad (5)$$

## III. ERROR MEASURES AND THEIR RESTRICTION TO QUBIT OBSERVABLES

We shall briefly consider the noise measure [6] and the metric error [3], and then specify them to the case of qubit systems. These measures are designed to be extensions of the root-mean square deviation used in statistics, but there exist some significant differences between them.

The noise measure is a state-dependent measure based on value comparison [1]. Suppose that we approximate a sharp observable  $A$ , referred to herein as a *target observable*, defined on  $\mathcal{H}$ , with first moment operator  $A = A[1]$ , via a measurement scheme  $\langle \mathcal{K}, \xi, U, Z \rangle$ , where  $\mathcal{K}$  is an ancillary Hilbert state prepared in a fixed state  $\xi$ ,  $U : \mathcal{H} \otimes \mathcal{K} \rightarrow \mathcal{H} \otimes \mathcal{K}$  is a unitary transformation and  $Z$  is a sharp observable (known as the *pointer observable*) defined on  $\mathcal{K}$ . This measurement model defines the approximating POVM  $C$  on  $\mathcal{H}$  (with the same number of effects as  $Z$ ) by

$$\langle \psi | C(k) \psi \rangle = \langle \psi \otimes \xi | U^* (I \otimes Z(k)) U \psi \otimes \xi \rangle, \quad (6)$$

which is to hold for all  $\psi \in \mathcal{H}$ . Following Arthurs and Goodman [9], one defines the *noise operator* by

$$N_A := U^*(I \otimes Z[1])U - A \otimes I, \quad (7)$$

and from this the noise measure (or measurement noise)  $\varepsilon(C, A, \psi)$  is given by

$$\varepsilon(C, A, \psi)^2 = \langle \psi \otimes \xi | N_A^2 \psi \otimes \xi \rangle. \quad (8)$$

By making use of Eq. (6) one can quickly see that

$$\varepsilon(C, A, \psi)^2 = \langle (C[1] - A)^2 \rangle_\psi + \langle C[2] - C[1]^2 \rangle_\psi, \quad (9)$$

where we denote by  $\langle \dots \rangle_\psi$  the expectation value in the state  $\psi$ .

One can take this one further step to unravel the probabilistic meaning and motivation of  $\varepsilon(C, A, \psi)^2$ . For simplicity, we present this for the case of interest in the remainder of this paper. Consider two POVMs  $A : a_i \mapsto A_i$  and  $C : a_i \mapsto C_i$  defined on the same set  $\Omega = \{a_1, \dots, a_N\}$ . Then Eq. (9) may be recast as

$$\varepsilon(C, A, \psi)^2 = \sum_{ij} (a_i - a_j)^2 \text{Re} \langle \psi | A_i C_j \psi \rangle. \quad (10)$$

Here we see that a probabilistic interpretation of  $\varepsilon(C, A, \psi)$  as a root-mean-square deviation becomes possible only in cases where all  $A_i$  and  $C_j$  commute on the state  $\psi$ , where  $\text{Re} \langle \psi | A_i C_j \psi \rangle = \langle \psi | A_i C_j \psi \rangle \geq 0$ .

In contrast to the measurement noise, the metric error is a state-independent measure focused on distribution comparison, and therefore defined as a distance function on the relevant set of POVMs. For our observables of interest,  $A : a_i \mapsto A_i$  and  $C : a_i \mapsto C_i$ , the so-called probabilistic distance  $D$  [10] is

$$\begin{aligned} D(C, A) &= 2 \max_{X \subset \Omega} \sup_{\rho} \left| \sum_{a_i \in X} (\text{tr} [\rho A_i] - \text{tr} [\rho C_i]) \right| \\ &= 2 \max_X \left\| \sum_{a_i \in X} (A_i - C_i) \right\|, \end{aligned} \quad (11)$$

where the supremum is taken over all density operators  $\rho$ , and  $\|\cdot\|$  denotes the operator norm. For dichotomic observables  $A : \pm 1 \mapsto A_{\pm}$ ,  $C : \pm 1 \mapsto C_{\pm}$  the probabilistic distance reduces to

$$D(C, A) = 2\|A_+ - C_+\|. \quad (12)$$

In the case of symmetric qubit observables with effects  $A_{\pm} = \frac{1}{2}(I \pm \mathbf{a} \cdot \boldsymbol{\sigma})$ ,  $C_{\pm} = \frac{1}{2}(I \pm \mathbf{c} \cdot \boldsymbol{\sigma})$ , the probabilistic distance becomes

$$D(C, A) = \|\mathbf{a} - \mathbf{c}\|. \quad (13)$$

By comparison, the noise measure is evaluated as

$$\begin{aligned} \varepsilon(C, A, \psi)^2 &= \|\mathbf{a} - \mathbf{c}\|^2 + 1 - \|\mathbf{c}\|^2 \\ &= D(C, A)^2 + U(C)^2, \end{aligned} \quad (14)$$

where we have introduced the *unsharpness*  $U(C)$  of  $C$ ,

$$U(C)^2 = 1 - \|\mathbf{c}\|^2. \quad (15)$$

A point of note is that  $\varepsilon(C, A, \psi)$  is independent of the state considered, and this is a consequence of considering symmetric observables. This state independence remains true also when the definition of  $\varepsilon$  is extended to include density operators.

In what follows we consider the approximation of two incompatible sharp qubit observables  $A$  and  $B$  via the jointly measurable approximating observables  $C$  and  $D$ , as illustrated in Fig. 1. All of these observables are dichotomic symmetric POVMs with values  $\pm 1$ . Given that the noise measures are state-independent in the case of these observables, we shall use the shorthands  $\varepsilon_A := \varepsilon(C, A, \rho)$  and  $\varepsilon_B := \varepsilon(D, B, \rho)$ .

#### IV. THE HALL/BRANCIARD JOINT APPROXIMATION CONCEPT

Hall [7] proposed a concept of joint approximation for a pair of selfadjoint operators representing observables  $A, B$  in terms of functions  $f(M), g(M)$  of an arbitrary observable  $M$ . The question of how to characterize optimal choices of approximations led him to the introduction of the measurement noise in a way that complements Ozawa's approach. This method is then used by Branciard [5] to derive a measurement noise tradeoff relation that is stronger than Ozawa's; we shall reconcile it with the concept of joint measurability already given and bring attention to some conceptual issues this method possesses.

Consider a system and auxiliary system with respective Hilbert spaces  $\mathcal{H}$  and  $\mathcal{K}$ , with the auxiliary system in a fixed state  $\xi$ . We define the discrete set  $\Omega = \{m_1, \dots, m_N\}$  and the discrete sharp observable  $M : m_j \mapsto M_j$  (with  $m_j \in \Omega$ ) associated with the self-adjoint operator  $M = M[1] = \sum_{j=1}^N m_j M_j$ , all acting on  $\mathcal{H} \otimes \mathcal{K}$ . In this construction, it is this self-adjoint operator that is considered to be the observable, and not the PVM that describes it. As a result, in [5] the discussion on approximating two observables is in fact focussed on approximating the two self-adjoint operators associated with two incompatible spectral measures,  $A = A[1]$  and  $B = B[1]$ , and not on the PVMs themselves. This approximation is performed by applying two functions,  $f, g : \mathbb{R} \rightarrow \mathbb{R}$ , on  $M$  such that for each eigenvalue  $m_j$ ,  $f(m_j)$  approximates an eigenvalue of the first self-adjoint operator and  $g(m_j)$  approximates an eigenvalue of the second. This then leads to the two approximating self-adjoint operators

$$f(M) = \sum_{j=1}^N f(m_j) M_j, \quad g(M) = \sum_{j=1}^N g(m_j) M_j. \quad (16)$$

Note that  $f(M)$  and  $g(M)$  commute with each other.

This construction can be viewed as an example of a Naimark dilation of a discrete POVM  $E : m_j \mapsto E_j$  on  $\mathcal{H}$ , defined via

$$\langle \psi | E_j \psi \rangle = \langle \psi \otimes \xi | M_j \psi \otimes \xi \rangle, \quad (17)$$

which must hold for all  $\psi \in \mathcal{H}$ . We can construct the POVMs  $\mathbf{C}$  and  $\mathbf{D}$  that approximate the target observables  $\mathbf{A}$  and  $\mathbf{B}$  as follows: Let  $\Omega_1 = f(\Omega) = \{c_1, \dots, c_n\}$  and  $\Omega_2 = g(\Omega) = \{d_1, \dots, d_m\}$ , then

$$\begin{aligned} \mathbf{C}(c_k) = C_k &= \sum_{m \in f^{-1}(c_k)} \mathbf{E}(m), \\ \mathbf{D}(d_\ell) = D_\ell &= \sum_{m \in g^{-1}(d_\ell)} \mathbf{E}(m), \end{aligned} \quad (18)$$

where  $f^{-1}(c_k) = \{m \in \Omega | f(m) = c_k\}$ , etc. In other words, the functions  $f$  and  $g$  induce partitions of the set of effects of  $\mathbf{M}$  and  $\mathbf{E}$ ; the bins of these partitions will necessarily overlap to some extent and, by including null effects when needed, we may define a joint observable  $\mathbf{J}$  for  $\mathbf{C}$  and  $\mathbf{D}$  as

$$\mathbf{J}(c_k, d_\ell) = \sum_m \chi_{f^{-1}(c_k) \cap g^{-1}(d_\ell)}(m) \mathbf{E}(m), \quad (19)$$

where  $\chi$  denotes the characteristic function. The positivity and normalization of these effects is given, and we quickly see that by summing over the outcome space of  $\mathbf{D}$  or  $\mathbf{C}$  we retrieve effects of  $\mathbf{C}$  or  $\mathbf{D}$ , respectively.

Whilst we have shown that recovering these observables as margins of  $\mathbf{J}$  is indeed possible, we must stress

that this method of deriving a joint approximation for two self-adjoint operators is physically lacking. This method was previously given in [7] based on the idea that “any measurement [in the sense presented above] is considered to provide a joint measurement of any two observables” if one simply rescales the values of the possible measurement outcomes via the functions  $f$  and  $g$ . However, as we show next, such functions can lead to suboptimal approximating observables that nevertheless possessing noise measure values equal to zero.

## V. SUBOPTIMAL APPROXIMATIONS WITH ZERO NOISE MEASURE VALUE

In [7], a derivation is given of the optimal post-selection functions  $f$  and  $g$  so that  $A = \mathbf{A}[1]$  and  $B = \mathbf{B}[1]$  are approximated via  $f(M)$  and  $g(M)$  with minimal noise measure, respectively. At the level of the system the approximating observables can be expressed in terms of the POVM  $\mathbf{E}$  via Eq. (18). The  $n^{\text{th}}$  moment operator  $\mathbf{C}[n]$  is then given by

$$\mathbf{C}[n] = \sum_k c_k^n \mathbf{C}(c_k) = \sum_m f(m)^n \mathbf{E}(m). \quad (20)$$

By expanding the form of the noise measure given in Eq. (9), where  $\psi$  is replaced by the density operator  $\rho$  and the expectation value by the trace, we find

$$\begin{aligned} \varepsilon(\mathbf{C}, \mathbf{A}, \rho)^2 &= \text{tr} [(\mathbf{C}[2] - \mathbf{C}[1]A - A\mathbf{C}[1] + A^2)\rho] \\ &= \sum_{c_k} c_k^2 \text{tr} [\mathbf{C}(c_k)\rho] - \sum_{c_k} c_k \text{tr} [(\mathbf{C}(c_k)A + A\mathbf{C}(c_k))\rho] + \text{tr} [A^2\rho] \\ &= \sum_m f(m)^2 \text{tr} [\mathbf{E}(m)\rho] - \sum_m f(m) \text{tr} [(\mathbf{E}(m)A + A\mathbf{E}(m))\rho] + \text{tr} [A^2\rho] \\ &= \sum_m \text{tr} [\mathbf{E}(m)\rho] \left( f(m) - \frac{\text{tr} [(\mathbf{E}(m)A + A\mathbf{E}(m))\rho]}{2 \text{tr} [\mathbf{E}(m)\rho]} \right)^2 - \sum_m \frac{\text{tr} [(\mathbf{E}(m)A + A\mathbf{E}(m))\rho]^2}{4 \text{tr} [\mathbf{E}(m)\rho]} + \text{tr} [A^2\rho]. \end{aligned} \quad (21)$$

The first term is non-negative and the last two terms are independent of  $f(m)$ . Therefore, the noise measure is minimized by setting

$$f(m) = \frac{\text{tr} [(\mathbf{E}(m)A + A\mathbf{E}(m))\rho]}{2 \text{tr} [\mathbf{E}(m)\rho]}. \quad (22)$$

However, just because a minimizing function  $f$  can be found, this does not mean that the observable  $\mathbf{C}$  defined by it is accurately approximating the observable  $\mathbf{A}$ . To illustrate this, we adopt an example given in [1]. Consider the observable  $\mathbf{A}$  associated with the self-adjoint operator

$$A = \mathbf{A}[1] = \frac{\gamma}{2}(\sigma_1 - \sigma_2), \quad (23)$$

where  $\sigma_1, \sigma_2$  denote the standard Pauli matrices, and  $\gamma = 2 - \sqrt{2}$ , for which we note the identity  $\gamma = \sqrt{2}(1 - \gamma)$ . The

observable  $\mathbf{E}$  that we shall measure, and whose outcomes we shall post-select to approximate  $\mathbf{A}$ , is a 3-outcome observable with effects

$$\begin{aligned} \mathbf{E}(1) &= \frac{\gamma}{2}(I + \sigma_1), \\ \mathbf{E}(2) &= \frac{\gamma}{2}(I + \sigma_2), \\ \mathbf{E}(3) &= 2(1 - \gamma)\frac{1}{2} \left( I - \frac{1}{\sqrt{2}}(\sigma_1 + \sigma_2) \right). \end{aligned} \quad (24)$$

These are all positive rank-one operators, and  $\mathbf{E}(1) + \mathbf{E}(2) + \mathbf{E}(3) = I$ . This observable is neither dichotomic nor symmetric, so the noise measure retains its state-

dependence, and we choose to measure in the state

$$\rho = \frac{1}{2} \left( I - \frac{1}{\sqrt{2}}(\sigma_1 + \sigma_2) \right). \quad (25)$$

With this choice of state we find the optimal post-selection function (22) to be of the form

$$f(1) = 1, \quad f(2) = -1, \quad f(3) = 0. \quad (26)$$

With these values, we find that the approximating observable  $C$  defined in Eq. (18) satisfies

$$C[1] = \sum_m f(m) E(m) = E(1) - E(2) = A, \quad (27a)$$

$$C[1]^2 = A^2 = \frac{\gamma^2}{2} I, \quad (27b)$$

$$C[2] = E(1) + E(2) = I - E(3), \quad (27c)$$

$$C[2] - C[1]^2 = 2(1 - \gamma) \frac{1}{2} \left( I + \frac{1}{\sqrt{2}}(\sigma_1 + \sigma_2) \right). \quad (27d)$$

Hence, the noise measure is equal to  $\varepsilon(C, A, \rho)^2 = 0$ , but in no way does  $C$  provide an accurate approximation of  $A$ , with the probability distributions  $p_\rho^A$  and  $p_\rho^C$  being very different:

$$p_\rho^A(\pm) = \frac{1}{2}, \quad p_\rho^C(\pm) = \frac{\gamma^2}{4}, \quad p_\rho^C(0) = 2(1 - \gamma). \quad (28)$$

It is interesting to see what  $\varepsilon(C, A, \rho)^2 = 0$  means in the case of approximators  $C$  that commute with  $A$ . In this case we can indeed characterize the entire class of “perfect” approximators. We begin by considering biased dichotomic observables, given by

$$C_\pm = \frac{(1 \pm \gamma)I \pm \mathbf{c} \cdot \boldsymbol{\sigma}}{2}, \quad (3)$$

where we require  $|\gamma| + \|\mathbf{c}\| \leq 1$  to ensure the positivity of  $C_\pm$ . This observable satisfies the following properties:

$$\begin{aligned} C[1] &= \gamma I + \mathbf{c} \cdot \boldsymbol{\sigma}, \\ C[1]^2 &= (\gamma^2 + \|\mathbf{c}\|^2)I + 2\gamma \mathbf{c} \cdot \boldsymbol{\sigma}, \\ C[2] &= I. \end{aligned} \quad (29)$$

If we let  $\mathbf{r}$  denote the Bloch vector of the density operator  $\rho$  describing the state of the system considered, then the noise measure for  $C$  approximating the sharp observable  $A$  is equal to

$$\begin{aligned} \varepsilon(C, A, \rho)^2 &= \text{tr} [\rho(\gamma I + (\mathbf{c} - \mathbf{a}) \cdot \boldsymbol{\sigma})^2] \\ &\quad + \text{tr} [\rho((1 - \gamma^2 - \|\mathbf{c}\|^2)I - 2\gamma \mathbf{c} \cdot \boldsymbol{\sigma})] \\ &= \gamma^2 + \|\mathbf{c} - \mathbf{a}\|^2 + 1 - \gamma^2 - \|\mathbf{c}\|^2 \\ &\quad + 2\gamma \text{tr} [\rho(\mathbf{c} - \mathbf{a} - \mathbf{c}) \cdot \boldsymbol{\sigma}] \\ &= 2(1 - \mathbf{a} \cdot \mathbf{c}) - 2\gamma \mathbf{r} \cdot \mathbf{a} \\ &= 2(1 - \mathbf{a} \cdot (\mathbf{c} + \gamma \mathbf{r})), \end{aligned} \quad (30)$$

where in the penultimate line we made use of the normalization of  $\mathbf{a}$ . The noise measure is now state dependent,

and it is possible to find situations where it goes to zero despite the approximating observable  $C$  not equalling  $A$ .

Assuming  $\gamma > 0$ , and noting that  $\|\mathbf{c}\| + \gamma \leq 1$ , then  $\varepsilon(C, A, \rho)^2 = 0$  implies that  $\mathbf{c} + \gamma \mathbf{r} = \mathbf{a}$ , which can only be achieved by having  $\mathbf{r} = \mathbf{a}$  and  $\mathbf{c} = \|\mathbf{c}\| \mathbf{a}$  (and hence  $\|\mathbf{c}\| + \gamma = 1$ ). It follows that  $C_- = \|\mathbf{c}\| A_-$ .

As an extreme case, we can set  $\gamma = 1$ , and so  $C_+ = I$  and  $C_- = 0$ . This is a trivial observable that nonetheless acts as an “optimal” approximating observable for this case.

It can be shown that the measurement noise may be zero for an  $n$ -outcome approximator  $C$  (commuting with  $A$ ) only if we consider an eigenstate of  $A$ , say,  $A\psi_+ = \psi_+$ . In that case one of the effects must have outcome equal to 1 and be of the form  $C_+ = I - cA_-$ , with  $0 \leq c < 1$ , while all other effects are of the form  $C_k = c_k A_-$ , where  $\sum_k c_k = c$ ,  $c_k \geq 0$ . (Note that the values of the outcomes associated with the  $C_k$  are arbitrary.)

In the first example,  $C$  is a noncommutative POVM and the coincidence of its first moment operator with the target operator means nothing more than the first moments of the statistics of measurements of  $A$  and  $C$  coincide in all states. The choice of state made ensures that the noisiness of  $C$  remains unnoticed, but this does not allow the conclusion that one has performed a perfect (error-free) estimation of  $A$ .

In the second example  $C$  and  $A$  commute, and the vanishing of  $\varepsilon(C, A, \rho)$  in the state with Bloch vector  $\mathbf{a}$  correctly reflects the fact that  $C$  and  $A$  are perfectly correlated in that  $A$ -eigenstate. While the measurement noise does have a well defined operational meaning when  $A$  and  $C$  commute, we find here that in such cases vanishing noise is exclusive to  $A$  eigenstates. In those cases the outcome of a possible accurate measurement of  $A$  is known with certainty, and even a very poor measurement scheme, such as doing nothing other than declaring the known outcome, will lead to perfectly accurate results; such measurements are represented by trivial observables (where each effect is a multiple of  $I$ ).

In both examples,  $\varepsilon = 0$  occurs for approximating observables that are vastly different to the target observable. Examples of this “false positive” behavior (that is, a false indication of perfect approximation accuracy) were first pointed out in 2004 in [11]. Ozawa’s response was a modification of the definition of  $\varepsilon$  into a quantity  $\bar{\varepsilon}$  called *local uniform error* [12, 13]. This quantity is designed to ensure that in any state  $\rho$ , one has  $\bar{\varepsilon} = 0$  if and only if the measurement gives perfect correlation between the values of  $A$  and  $C$ . However, this quantity possesses the undesirable property of being discontinuous in the state variable, as shall be illustrated next.

For qubit observables, one has  $\bar{\varepsilon}^2 = \varepsilon^2$  if  $\rho$  is (the projection onto) an eigenstate of  $A$  and otherwise  $\bar{\varepsilon}^2$  is the state-independent quantity

$$\bar{\varepsilon}^2 = \max_{\rho'} \varepsilon(A, C, \rho')^2 = 2(1 - \mathbf{a} \cdot \mathbf{c} + |\gamma|),$$

where the maximum is over *all* states. In the first case, we have seen that indeed there are perfect correlations

between  $\mathbf{A}$  and even the trivial (“do nothing”) measurement, for which the statement  $\bar{\varepsilon} = 0$  adds no useful information. In the latter case,  $\bar{\varepsilon}^2 = 0$  implies that  $\mathbf{A} = \mathbf{C}$ —but one has lost the intended feature of state-dependence of the noise measure. Moreover, even a slight deviation of  $\rho$  from an eigenstate of  $\mathbf{A}$  leads to a discontinuous jump in the value of  $\bar{\varepsilon}^2$ : in the case of the trivial observable  $C_+ = I, C_- = 0$  (for which  $\gamma = 1$  and  $\|\mathbf{c}\| = 0$ ), this value then changes from 0 to 4.

## VI. ERROR BOUNDS: OPTIMAL APPROXIMATIONS

We now turn to the determination of the optimal error bounds, by specifying the lower boundary of the admissible region of pairs of error values that can be attained through a joint measurement of  $\mathbf{C}$  and  $\mathbf{D}$  as approximators to  $\mathbf{A}$  and  $\mathbf{B}$ , respectively [14].

### A. Bound for metric error

In [2] it was shown that the optimal compatible approximators  $\mathbf{C}, \mathbf{D}$  for  $\pm 1$  valued qubit observables  $\mathbf{A}, \mathbf{B}$  are to be found among the symmetric qubit POVMs characterized above. For these, the compatibility condition is simply described by (5),

$$\|\mathbf{c} + \mathbf{d}\| + \|\mathbf{c} - \mathbf{d}\| \leq 2. \quad (31)$$

The optimization consists of finding the smallest possible value of  $D(\mathbf{D}, \mathbf{B}) = \|\mathbf{b} - \mathbf{d}\|$  for a given value of  $D(\mathbf{C}, \mathbf{A}) = \|\mathbf{a} - \mathbf{c}\|$ . Thus, to be specific, one restricts the class of compatible approximators  $\mathbf{C}, \mathbf{D}$  to those with a fixed given value of  $D(\mathbf{C}, \mathbf{A})$ , and must identify the pairs  $\mathbf{C}, \mathbf{D}$  for which  $D(\mathbf{D}, \mathbf{B})$  is then minimal. This problem was solved in [4]. We briefly summarize the construction here, providing some detail and steps that were not given explicitly in that paper.

The geometric significance of the constraint (31) is readily determined: if we fix the vector  $\mathbf{c}$ , say, then this equation defines an ellipsoid within which the end points of all possible vectors  $\mathbf{d}$  must lie for the corresponding observable  $\mathbf{D}$  to be jointly measurable with  $\mathbf{C}$ . Similarly, there is an “admissible” ellipse for the vectors  $\mathbf{c}$  for any fixed  $\mathbf{d}$ .

By fixing  $\mathbf{c}$  and varying  $\mathbf{d}$ , one needs to identify the smallest circle centred at (the end point of)  $\mathbf{b}$  that will intersect with the ellipsoid of allowed vectors  $\mathbf{d}$  described by Eq. (31). By using the method of Lagrange multipliers to minimize the radius  $D(\mathbf{D}, \mathbf{B})$  subject to inequality (31) we find that the end point of  $\mathbf{d}$  lies on the surface of the ellipsoid, and the vector  $\mathbf{b} - \mathbf{d}$  is normal to that surface. The optimization is complete only when  $\mathbf{c}$  is also found simultaneously so that its end point lies on the surface of the corresponding ellipsoid of “allowed” vectors  $\mathbf{c}$  for the given  $\mathbf{d}$ , with  $\mathbf{a} - \mathbf{c}$  normal to that surface.

In this case, we have  $\|\mathbf{c} + \mathbf{d}\| + \|\mathbf{c} - \mathbf{d}\| = 2$ , which can be equivalently expressed in the form

$$\|\mathbf{c}\|^2 + \|\mathbf{d}\|^2 = 1 + (\mathbf{c} \cdot \mathbf{d})^2 = 1 + M^2, \quad (32)$$

where  $M = \mathbf{c} \cdot \mathbf{d}$ . Another equivalent form of this condition is  $\|\mathbf{c} \pm \mathbf{d}\| = 1 \pm M$ . We may therefore introduce the unique angle  $\varphi$  satisfying

$$\sin \varphi = \sqrt{\frac{1 - \|\mathbf{d}\|^2}{1 - M^2}}, \quad \cos \varphi = \sqrt{\frac{1 - \|\mathbf{c}\|^2}{1 - M^2}}. \quad (33)$$

The optimality condition for  $\mathbf{c}, \mathbf{d}$  described above is then characterized by the simultaneous Euler-Lagrange equations, which can be brought into the forms

$$\begin{aligned} \mathbf{a} - \mathbf{c} &= \frac{\mu D(\mathbf{C}, \mathbf{A})(\mathbf{c} - M\mathbf{d})}{(1 - M^2) \sin \varphi}, \\ \mathbf{b} - \mathbf{d} &= \frac{\nu D(\mathbf{D}, \mathbf{B})(\mathbf{d} - M\mathbf{c})}{(1 - M^2) \cos \varphi}. \end{aligned} \quad (34)$$

where  $\mu = \pm 1, \nu = \pm 1$ . We note that  $\mathbf{a} - \mathbf{c}$  and  $\mathbf{b} - \mathbf{d}$  are mutually orthogonal as a consequence of Eq. (32).

Recalling that  $\|\mathbf{a}\|^2 = \|\mathbf{b}\|^2 = 1$ , we find that

$$\begin{aligned} D(\mathbf{C}, \mathbf{A}) &= \sqrt{1 - M^2 \cos^2 \varphi} - \mu \sin \varphi, \\ D(\mathbf{D}, \mathbf{B}) &= \sqrt{1 - M^2 \sin^2 \varphi} - \nu \cos \varphi. \end{aligned} \quad (35)$$

As  $M^2 \in [0, 1]$ , Eq. (35) guarantees that  $D(\mathbf{C}, \mathbf{A})$  and  $D(\mathbf{D}, \mathbf{B})$  are indeed non-negative. The combinations  $\mu = \nu = 1$  and  $\mu = \nu = -1$  correspond to simultaneous minima and maxima for  $D(\mathbf{C}, \mathbf{A})$  and  $D(\mathbf{D}, \mathbf{B})$ , respectively, while the conditions  $\mu = 1 = -\nu, -\mu = 1 = \nu$  correspond to mixed cases (see Fig. 2). Here we focus on the minimum, hence putting  $\mu = \nu = 1$ .

An expression for  $M$  can be obtained by working out  $\cos \theta := \mathbf{a} \cdot \mathbf{b}$  using (34). After some repeated squaring and elimination of terms one obtains:

$$(\cos^2 \theta - M^2)^2 = M^4 \sin^2 2\varphi \sin^2 \theta. \quad (36)$$

In the evaluation of  $\mathbf{a} \cdot \mathbf{b}$  one finds that

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{c} \cdot \mathbf{d} + \text{nonnegative terms},$$

so that (recalling that  $M = \mathbf{c} \cdot \mathbf{d}$ ) Eq. (36) gives the unique solution

$$M^2 = \frac{\cos^2 \theta}{1 + \sin \theta \sin 2\varphi}. \quad (37)$$

Putting the expression for  $M^2$  into Eqs. (35) and using the identity  $1 = \sin^2 \varphi + \cos^2 \varphi (\cos^2 \theta + \sin^2 \theta)$  we find the minimum error bounds to be

$$\begin{aligned} D(\mathbf{C}, \mathbf{A}) &= \frac{\sin \varphi + \sin \theta \cos \varphi}{\sqrt{1 + \sin \theta \sin 2\varphi}} - \sin \varphi, \\ D(\mathbf{D}, \mathbf{B}) &= \frac{\cos \varphi + \sin \theta \sin \varphi}{\sqrt{1 + \sin \theta \sin 2\varphi}} - \cos \varphi. \end{aligned} \quad (38)$$

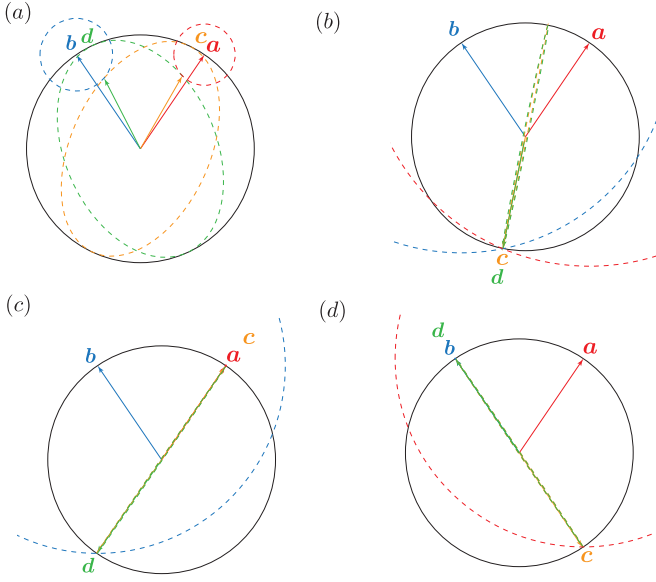


FIG. 2. Examples of vectors  $\mathbf{c}$  and  $\mathbf{d}$  described by Eq. (34) in the case of (a)  $\mu = \nu = 1$ , (b)  $\mu = \nu = -1$ , (c)  $\mu = 1 = -\nu$ , (d)  $\mu = -1 = -\nu$ .

We next provide an operational interpretation of this solution and the relevant quantities involved. Using the unsharpness measures,  $U(\mathbf{C})^2 = 1 - \|\mathbf{c}\|^2$ ,  $U(\mathbf{D})^2 = 1 - \|\mathbf{d}\|^2$  (see Eq. (15)), we can rewrite the compatibility equation (32) in the equivalent forms

$$1 - M^2 = U(\mathbf{C})^2 + U(\mathbf{D})^2, \quad (39)$$

$$U(\mathbf{C})^2 U(\mathbf{D})^2 = \|\mathbf{c} \times \mathbf{d}\|^2 = 4 \|[C_+, D_+]\|^2. \quad (40)$$

The latter equation shows that the compatibility of  $\mathbf{C}$ ,  $\mathbf{D}$  requires these observables to be sufficiently unsharp if they do not commute with each other.

Using the above relations, we can determine  $\varphi$  solely in terms of the unsharpness of the two approximating observables:

$$\sin \varphi = \frac{U(\mathbf{D})}{\sqrt{U(\mathbf{C})^2 + U(\mathbf{D})^2}},$$

$$\cos \varphi = \frac{U(\mathbf{C})}{\sqrt{U(\mathbf{C})^2 + U(\mathbf{D})^2}},$$

and further

$$\sin 2\varphi = \frac{2U(\mathbf{C})U(\mathbf{D})}{U(\mathbf{C})^2 + U(\mathbf{D})^2}.$$

Substituting these identities in Eq. (38) we can express the bounds in terms of the unsharpness measures of  $\mathbf{C}$  and  $\mathbf{D}$  and the degree of noncommutativity of  $A = A[1] = \mathbf{a} \cdot \boldsymbol{\sigma}$  and  $B = B[1] = \mathbf{b} \cdot \boldsymbol{\sigma}$ , given by

$$\sin^2 \theta = \|\mathbf{a} \times \mathbf{b}\|^2 = \frac{1}{4} \|[A, B]\|^2; \quad (41)$$

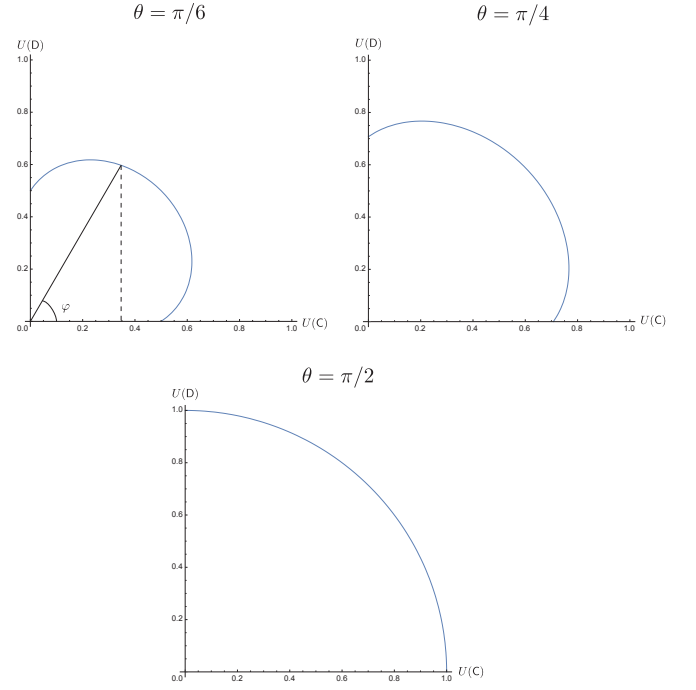


FIG. 3. How the tradeoff between  $U(\mathbf{C})$  and  $U(\mathbf{D})$  described by Eq. (43) varies with  $\theta$ . The angle  $\varphi$  is also highlighted.

we obtain:

$$D(\mathbf{C}, \mathbf{A}) = \frac{U(\mathbf{D}) + U(\mathbf{C}) \sin \theta}{\sqrt{(U(\mathbf{D}) + U(\mathbf{C}) \sin \theta)^2 + U(\mathbf{C})^2 \cos^2 \theta}} - \frac{U(\mathbf{D})}{\sqrt{U(\mathbf{C})^2 + U(\mathbf{D})^2}}, \quad (42)$$

$$D(\mathbf{D}, \mathbf{B}) = \frac{U(\mathbf{C}) + U(\mathbf{D}) \sin \theta}{\sqrt{(U(\mathbf{C}) + U(\mathbf{D}) \sin \theta)^2 + U(\mathbf{D})^2 \cos^2 \theta}} - \frac{U(\mathbf{C})}{\sqrt{U(\mathbf{C})^2 + U(\mathbf{D})^2}}.$$

Note that due to its relative ease of use, in what we will be considering we shall make use of the form of the bound given in Eq. (38).

We can find an implicit functional relation between  $U(\mathbf{C})$  and  $U(\mathbf{D})$  by rewriting Eq. (37) with  $\sin \theta$  as the subject:

$$\sin \theta = -\frac{1}{2} M^2 \sin 2\varphi + \sqrt{\frac{1}{4} M^4 \sin^2 2\varphi + (1 - M^2)}. \quad (43)$$

We briefly consider two limiting cases (some further cases are considered in Fig. 3):

(a)  $\theta = 0$ ,  $A = B$ : Then  $M^2 = 1$ ,  $U(\mathbf{C}) = U(\mathbf{D}) = 0$ , and  $\mathbf{c} = \mathbf{d} = \mathbf{a} = \mathbf{b}$ .

(b)  $\theta = \frac{\pi}{2}$ : Then  $M^2 = 0$ ,  $U(\mathbf{C})^2 + U(\mathbf{D})^2 = 1$ ,  $\mathbf{c} = \|\mathbf{c}\| \mathbf{a}$ ,  $\mathbf{d} = \|\mathbf{d}\| \mathbf{b}$ . In general, the shape of the bound described by Eq. (38) is determined by the angle  $\theta$  that quantifies the incompatibility between  $A$  and  $B$ . In the case of compatible observables ( $\theta = 0$ ), the optimal errors are both

zero, as is expected, and for maximally incompatible observables ( $\theta = \pi/2$ ) we arrive at the form

$$D(C, A) = 1 - \sin \varphi, \quad D(D, B) = 1 - \cos \varphi, \quad (44)$$

which for the domain that we are considering,  $\varphi \in [0, \pi/2]$ , defines the lower left quadrant of a unit circle centred at  $(1, 1)$ :

$$(D(C, A) - 1)^2 + (D(D, B) - 1)^2 = 1. \quad (45)$$

For all other values of  $\theta$ , this bound defines a curve that lies slightly above a circle of radius  $\sin \theta$  centred at  $(\sin \theta, \sin \theta)$ . If we let  $\varphi = 0$  in Eq. (38) we see that  $D(C, A) = \sin \theta$  and  $D(D, B) = 0$ , whilst if  $\varphi = \pi/2$  we find  $D(C, A) = 0$  and  $D(D, B) = \sin \theta$ .

To conclude this discussion, we can use equations (37) and (38) to obtain a generalization of Eq. (45):

$$D(C, A) \sin \varphi + D(D, B) \cos \varphi = \frac{\cos \theta}{M} - 1. \quad (46)$$

In [4], a thorough discussion of the boundary curve of the admissible region of points  $(D(C, A), D(D, B))$  is given with  $\varphi$  as the parameter. In particular, the family of equations (46) is interpreted as providing tangential straight-line estimates to the lower boundary curve. Here we have added the interpretation in terms of the degrees of unsharpness of  $C$  and  $D$ , and established a functional relation between these parameters. We have seen that the optimal approximations strike a balance between the necessary degrees of unsharpness required to ensure compatibility and the magnitude of the approximation errors. In other words, a certain degree of noncommutativity of  $C, D$  is usually favorable for improving the approximations, but too much noncommutativity requires more unsharpness (for compatibility), which drives the errors up again.

### B. Minimizing measurement noise for compatible approximations

Our next task is to determine the lower boundary curve in the admissible region of the measurement noise pairs,  $(\varepsilon_A, \varepsilon_B)$ , under the constraint of compatibility for the approximators  $C, D$ .

Thus, fixing a Bloch vector  $\mathbf{c}$ , we wish to determine the the smallest  $\varepsilon_B^2 = 2(1 - \mathbf{b} \cdot \mathbf{d})$  subject to  $\mathbf{c}$  and  $\mathbf{d}$  satisfying inequality (31). Optimality is again to be obtained by vectors  $\mathbf{c}, \mathbf{d}$  pointing to the surfaces of their respective compatibility ellipsoids, thus saturating (31).

Using the Lagrange multiplier method, we now find that  $\mathbf{a}$  and  $\mathbf{b}$  are of the form

$$\begin{aligned} \mathbf{a} &= \frac{\mathbf{c} - M\mathbf{d}}{(1 - M^2) \sin \varphi}, \\ \mathbf{b} &= \frac{\mathbf{d} - M\mathbf{c}}{(1 - M^2) \cos \varphi}, \end{aligned} \quad (47)$$

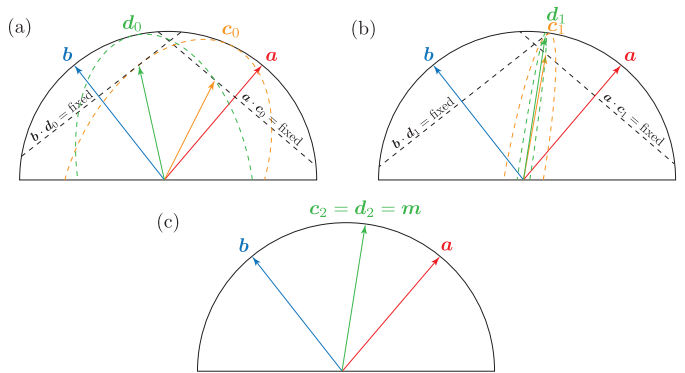


FIG. 4. Why a local minimum for the noise measure cannot be found. (a) Assume that a vector  $\mathbf{c}_0$  is chosen to approximate  $\mathbf{a}$ , giving a noise measure value of  $2(1 - \mathbf{a} \cdot \mathbf{c}_0)$  and an ellipse of jointly measurable vectors, shown here by the orange dashed ellipse segment (color version online). The optimal vector  $\mathbf{d}_0$  approximating  $\mathbf{b}$  is such that  $\mathbf{d}_0 - (\mathbf{b} \cdot \mathbf{d}_0)\mathbf{b}$  is tangent to the ellipse. However, by considering the ellipse surrounding  $\mathbf{d}_0$  (the green dashed ellipse segment) we see that  $\mathbf{c}_0$  is not optimal. (b) We now replace  $\mathbf{c}_0$  by  $\mathbf{c}_1$ , only to find that  $\mathbf{d}_1$  is no longer optimal. (c) We repeat this iterative process of finding optimal  $\mathbf{d}$  (resp.  $\mathbf{c}$ ) given a vector  $\mathbf{c}$  (resp.  $\mathbf{d}$ ) until we reach the normalized vector  $\mathbf{m}$ . Whilst we are able to find a minimum, at each step we are pushed to a new optimizing vector, and so the two optimizing vectors will not agree unless they are both already normalized.

with  $M$  and  $\varphi$  of the form given above. In view of (32), the vectors of the right-hand sides are mutually orthogonal, so that the Lagrange multiplier method only gives extrema in the case of orthogonal  $\mathbf{a}, \mathbf{b}$ .

In other words, there is no local minimum unless the target observables  $A$  and  $B$  are maximally incompatible. In order to see this, consider Fig. 4. If we fix a vector  $\mathbf{c}_0$  to approximate  $\mathbf{a}$ , then we define an ellipse of possible  $\mathbf{d}$  that are jointly measurable with  $\mathbf{c}_0$ , with the vector  $\mathbf{d}_0$  that minimizes the noise value  $\varepsilon_B^2$  being such that the line  $\mathbf{d}_0 - (\mathbf{b} \cdot \mathbf{d}_0)\mathbf{b}$  is tangent to the ellipse at  $\mathbf{d}_0$ . However, if we were to find the vector  $\mathbf{c}_1$  that minimizes the noise value  $\varepsilon_A^2$  subject to joint measurability with  $\mathbf{d}_0$  we would find that our solution would not be  $\mathbf{c}_0$ . Indeed, at no point are we able to find a pair of vectors  $\mathbf{c}$  and  $\mathbf{d}$  such that  $\mathbf{c}$  minimizes  $\varepsilon_A^2$  and  $\mathbf{d}$  minimizes  $\varepsilon_B^2$  simultaneously subject to Eq. (31). As we see in Fig. 4, by iteratively minimizing the noise measures we reduce to the case of a normalized vector  $\mathbf{m}$ , confirming what is found by Branciard in the qubit case [5].

We may use this observation to provide an alternative proof of the bound for the noise measure in the qubit case, as given by Branciard. The original proof was given for any finite-dimensional Hilbert space and is reliant on the state-dependent quantity  $C_{AB} = \frac{1}{2i} \langle [A, B] \rangle_\psi$ . The proof that we give is state-independent, which is appropriate given that we are dealing with noise measure values that are themselves state-independent for the approximating observables used.



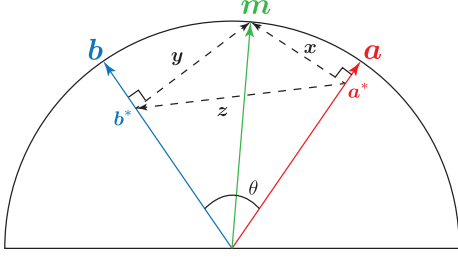


FIG. 5. A Bloch-geometric derivation of Eq. (59).

The outline for our proof is given in Fig. 5. We begin by choosing a normalized vector  $\mathbf{m}$  that, by the preceding argument, we know minimizes the noise value when approximating  $\mathbf{a}$  and  $\mathbf{b}$ . We now rewrite  $\mathbf{m}$  in the following way:

$$\mathbf{m} = \mathbf{a}^* + \mathbf{x} = \mathbf{b}^* + \mathbf{y}, \quad (48)$$

where  $\mathbf{a}^* = (\mathbf{a} \cdot \mathbf{m}) \mathbf{a}$  and  $\mathbf{b}^* = (\mathbf{b} \cdot \mathbf{m}) \mathbf{b}$ . By construction,  $\mathbf{x} \perp \mathbf{a}$  and  $\mathbf{y} \perp \mathbf{b}$ , and from Eq. (48) we can define the vector  $\mathbf{z} = \mathbf{b}^* - \mathbf{a}^* = \mathbf{x} - \mathbf{y}$ . The square of the norm of  $\mathbf{z}$  is therefore

$$\|\mathbf{z}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\mathbf{x} \cdot \mathbf{y}. \quad (49)$$

By rearranging Eq. (48) we can find the norms of  $\mathbf{x}$  and  $\mathbf{y}$ :

$$\begin{aligned} \|\mathbf{x}\|^2 &= \|\mathbf{m} - \mathbf{a}^*\|^2 = 1 - (\mathbf{a} \cdot \mathbf{m})^2 = \varepsilon_A^2 \left(1 - \frac{\varepsilon_A^2}{4}\right), \\ \|\mathbf{y}\|^2 &= \|\mathbf{m} - \mathbf{b}^*\|^2 = 1 - (\mathbf{b} \cdot \mathbf{m})^2 = \varepsilon_B^2 \left(1 - \frac{\varepsilon_B^2}{4}\right), \end{aligned} \quad (50)$$

and by geometric considerations, we see that

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} &= \|\mathbf{x}\| \|\mathbf{y}\| \cos(\pi - \theta) \\ &= -\varepsilon_A \varepsilon_B \sqrt{1 - \frac{\varepsilon_A^2}{4}} \sqrt{1 - \frac{\varepsilon_B^2}{4}} \cos \theta, \end{aligned} \quad (51)$$

where  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$ . Hence,

$$\begin{aligned} \|\mathbf{z}\|^2 &= \varepsilon_A^2 \left(1 - \frac{\varepsilon_A^2}{4}\right) + \varepsilon_B^2 \left(1 - \frac{\varepsilon_B^2}{4}\right) \\ &\quad + 2\varepsilon_A \varepsilon_B \sqrt{1 - \frac{\varepsilon_A^2}{4}} \sqrt{1 - \frac{\varepsilon_B^2}{4}} \cos \theta. \end{aligned} \quad (52)$$

In order to evaluate the left hand side of Eq. (52) we make use of the form  $\mathbf{z} = \mathbf{b}^* - \mathbf{a}^*$ :

$$\begin{aligned} \|\mathbf{z}\|^2 &= \|\mathbf{b}^* - \mathbf{a}^*\|^2 \\ &= (\mathbf{a} \cdot \mathbf{m})^2 + (\mathbf{b} \cdot \mathbf{m})^2 - 2(\mathbf{a} \cdot \mathbf{m})(\mathbf{b} \cdot \mathbf{m})(\mathbf{a} \cdot \mathbf{b}) \\ &= 1 - (1 - (\mathbf{a} \cdot \mathbf{m})^2)(1 - (\mathbf{b} \cdot \mathbf{m})^2) \\ &\quad + ((\mathbf{a} \cdot \mathbf{m})(\mathbf{b} \cdot \mathbf{m}) - (\mathbf{a} \cdot \mathbf{b}))^2 - (\mathbf{a} \cdot \mathbf{b})^2. \end{aligned} \quad (53)$$

By making use of the identity

$$(\mathbf{e} \times \mathbf{f}) \cdot (\mathbf{g} \times \mathbf{h}) = (\mathbf{e} \cdot \mathbf{g})(\mathbf{f} \cdot \mathbf{h}) - (\mathbf{e} \cdot \mathbf{h})(\mathbf{f} \cdot \mathbf{g}), \quad (54)$$

for  $\mathbf{e}, \mathbf{f}, \mathbf{g}, \mathbf{h} \in \mathbb{R}^3$ , we see that

$$(\mathbf{a} \cdot \mathbf{m})(\mathbf{b} \cdot \mathbf{m}) - (\mathbf{a} \cdot \mathbf{b}) = (\mathbf{m} \times \mathbf{a}) \cdot (\mathbf{m} \times \mathbf{b}), \quad (55)$$

and so

$$\begin{aligned} \|\mathbf{z}\|^2 &= 1 - (1 - (\mathbf{a} \cdot \mathbf{m})^2)(1 - (\mathbf{b} \cdot \mathbf{m})^2) \\ &\quad + ((\mathbf{m} \times \mathbf{a}) \cdot (\mathbf{m} \times \mathbf{b}))^2 - (\mathbf{a} \cdot \mathbf{b})^2. \end{aligned} \quad (56)$$

Since  $\mathbf{m}, \mathbf{a}$  and  $\mathbf{b}$  span the same plane, the vectors  $\mathbf{m} \times \mathbf{a}$  and  $\mathbf{m} \times \mathbf{b}$  are collinear, hence

$$\begin{aligned} ((\mathbf{m} \times \mathbf{a}) \cdot (\mathbf{m} \times \mathbf{b}))^2 &= \|\mathbf{m} \times \mathbf{a}\|^2 \|\mathbf{m} \times \mathbf{b}\|^2 \\ &= (1 - (\mathbf{a} \cdot \mathbf{m})^2)(1 - (\mathbf{b} \cdot \mathbf{m})^2), \end{aligned} \quad (57)$$

which reduces the square of the norm of  $\mathbf{z}$  to

$$\|\mathbf{z}\|^2 = 1 - (\mathbf{a} \cdot \mathbf{b})^2 = \|\mathbf{a} \times \mathbf{b}\|^2 = \sin^2 \theta. \quad (58)$$

In other words, we have reached the bound

$$\begin{aligned} \varepsilon_A^2 \left(1 - \frac{\varepsilon_A^2}{4}\right) + \varepsilon_B^2 \left(1 - \frac{\varepsilon_B^2}{4}\right) \\ + 2\varepsilon_A \varepsilon_B \cos \theta \sqrt{\left(1 - \frac{\varepsilon_A^2}{4}\right) \left(1 - \frac{\varepsilon_B^2}{4}\right)} \geq \sin^2 \theta, \end{aligned} \quad (59)$$

where the inequality comes from the requirement that the considered vector  $\mathbf{m}$  is a minimizing case. This is the bound given by Branciard for the noise measure in the qubit case [15]. Our approach is based directly on establishing the minimal error of one approximator given the error of the other.

## VII. COMPARISON OF OPTIMAL APPROXIMATORS

In their respective papers, Branciard and Yu and Oh provide examples of approximating observables  $\mathbf{C}$  and  $\mathbf{D}$  that saturate their given bounds. In this section we shall compare these optimal approximators and find that they (almost always) comprise disjoint sets.

We will also discuss the experimental work by Ringbauer *et al.* [16], who have constructed jointly measurable approximators of two maximally incompatible observables whose measurement noise values are close to the bound given in Eq. (59). Their method is similar in nature to that presented in [5], and we will see that the metric error values for these approximators are not close to the bound given by Eq. (38). This is in contrast with the experiment of Rozema *et al.* [17], which does lead to the optimal bound for  $D(\mathbf{C}, \mathbf{A})$  and  $D(\mathbf{D}, \mathbf{B})$  when  $\mathbf{A}$  and  $\mathbf{B}$  are maximally incompatible [3, 18].

### A. Optimal approximators with respect to the noise measure

We shall begin with the optimal approximators identified by Branciard with respect to the noise measure. The

sharp target observables to be approximated are represented by Bloch unit vectors  $\mathbf{a}$  and  $\mathbf{b}$  that lie within the  $x$ - $y$  plane of the Bloch sphere; they are characterized by the angles  $\phi_a$  and  $\phi_b$ , respectively:

$$\begin{aligned} A &:= \mathbf{A}[1] = \cos \phi_a \sigma_1 + \sin \phi_a \sigma_2, \\ B &:= \mathbf{B}[1] = \cos \phi_b \sigma_1 + \sin \phi_b \sigma_2, \end{aligned} \quad (60)$$

where it is assumed that  $\phi_a \leq \phi_b$ , and we identify  $\theta = \phi_b - \phi_a$ .

The optimal approximator for these observables in terms of the noise measure is the sharp dichotomic observable  $\mathbf{M}$  with an associated first moment operator  $M$  of the form

$$M = \cos \phi \sigma_1 + \sin \phi \sigma_2 = \mathbf{m} \cdot \boldsymbol{\sigma}, \quad (61)$$

where  $\phi \in [\phi_a, \phi_b]$ .

In other words, as mentioned in Section VIB, what is measured here is a single sharp observable as an approximator for both  $\mathbf{A}$  and  $\mathbf{B}$ , with its respective Bloch vector  $\mathbf{m}$  lying somewhere in-between  $\mathbf{a}$  and  $\mathbf{b}$  on the circumference of the circle in the plane spanned by the two vectors. In the language of the joint measurement scheme discussed in Section IV, this is done by setting  $f(m) = g(m) = m$  for all  $m$  within the spectrum of  $M$ . The value of the noise measures  $\varepsilon_A^2$  and  $\varepsilon_B^2$  are given via Eq. (14):

$$\begin{aligned} \varepsilon_A^2 &= \|\mathbf{a} - \mathbf{m}\|^2 = 2(1 - \mathbf{a} \cdot \mathbf{m}) \\ &= 2(1 - \cos(\phi - \phi_a)) \\ &= 4 \sin^2 \left( \frac{\phi - \phi_a}{2} \right), \\ \varepsilon_B^2 &= \|\mathbf{b} - \mathbf{m}\|^2 = 4 \sin^2 \left( \frac{\phi_b - \phi}{2} \right). \end{aligned} \quad (62)$$

From this the expression  $\varepsilon_A^2(1 - \varepsilon_A^2/4)$  is given by

$$\begin{aligned} \varepsilon_A^2 \left( 1 - \frac{\varepsilon_A^2}{4} \right) &= 4 \sin^2 \left( \frac{\phi - \phi_a}{2} \right) \cos^2 \left( \frac{\phi - \phi_a}{2} \right) \\ &= \sin^2(\phi - \phi_a), \end{aligned} \quad (63)$$

and similarly for  $\varepsilon_B^2(1 - \varepsilon_B^2/4)$ . Making use of the above values for  $\varepsilon_A^2(1 - \varepsilon_A^2/4)$  and  $\varepsilon_B^2(1 - \varepsilon_B^2/4)$ , whilst noting that  $\phi_b = \phi_a + \theta$ , the left hand side of Eq. (59) is found in this case to be equal to

$$\begin{aligned} \sin^2(\phi - \phi_a) + \sin^2(\phi_b - \phi) \\ + 2 \cos \theta \sin(\phi - \phi_a) \sin(\phi_b - \phi) = \sin^2 \theta. \end{aligned} \quad (64)$$

In other words, for any value of  $\phi_a$  and  $\phi_b$ , where  $\phi_a \leq \phi_b$ , and any  $\phi \in [\phi_a, \phi_b]$ , the sharp observable  $\mathbf{M}$ , whose first moment operator is given by Eq. (61), approximates both  $\mathbf{A}$  and  $\mathbf{B}$  simultaneously to such a degree that the noise measure quantities  $\varepsilon_A$  and  $\varepsilon_B$  saturate the bound given in Eq. (59), i.e., this bound is saturated by any sharp observable whose normalized Bloch vector lies in-between  $\mathbf{a}$  and  $\mathbf{b}$  in the plane spanned by them (see Fig. 4).

Furthermore, suppose that we consider the unit vector  $\mathbf{m}$ , and then define the vectors  $\mathbf{c} = (\mathbf{a} \cdot \mathbf{m})\mathbf{a}$  and  $\boldsymbol{\lambda} = \lambda \mathbf{c} + (1 - \lambda)\mathbf{m}$ , where  $\lambda \in [0, 2]$  so that  $\|\boldsymbol{\lambda}\| \leq 1$ . In this case any dichotomic observable approximating  $\mathbf{A}$  described by the Bloch vector  $\boldsymbol{\lambda}$  satisfies

$$\begin{aligned} \varepsilon_A^2 &= \|\mathbf{a} - \boldsymbol{\lambda}\|^2 + 1 - \|\boldsymbol{\lambda}\|^2 \\ &= 2(1 - \mathbf{a} \cdot \boldsymbol{\lambda}) \\ &= 2(1 - \mathbf{a} \cdot \mathbf{m}). \end{aligned} \quad (65)$$

In other words, for any vector on the line segment between  $\mathbf{m}$  and  $\mathbf{c}$ , we find that the noise measure is fixed, although the level to which these vectors approximate  $\mathbf{A}$  varies.

In the case of maximally incompatible target observables—where  $\mathbf{a} \perp \mathbf{b}$ —we find that it is not just normalized vectors which optimize the noise measure, but any pair of smeared vectors  $\mathbf{a}^* = (\mathbf{a} \cdot \mathbf{m})\mathbf{a}$  and  $\mathbf{b}^* = (\mathbf{b} \cdot \mathbf{m})\mathbf{b}$ , with  $\|\mathbf{m}\| = 1$ , will also achieve the lower band in Eq. (59). Suppose that we define the vectors

$$\begin{aligned} \mathbf{c}_\lambda &:= \mathbf{a}^* + \lambda \mathbf{b}^*, \\ \mathbf{d}_\mu &:= \mathbf{b}^* + \mu \mathbf{a}^*. \end{aligned} \quad (66)$$

For any value of  $\lambda, \mu \in [0, 1]$  the noise measure has a fixed value, but there is no guarantee that these vectors will all be jointly measurable, as dictated by Eq. (31), or more usefully by Eq. (32). In the case  $\mathbf{c}_1 = \mathbf{d}_1 = \mathbf{m}$ , joint measurability is trivially satisfied, whilst for the cases of  $\mathbf{a}^* = \mathbf{c}_0$  and  $\mathbf{b}^* = \mathbf{d}_0$  we have  $\|\mathbf{a}^*\|^2 + \|\mathbf{b}^*\|^2 = \|\mathbf{m}\|^2 = 1$  by the Pythagorean Theorem (see Fig. 6), and since  $\mathbf{a}^* \perp \mathbf{b}^*$  we see that  $\|\mathbf{a}^*\|^2 + \|\mathbf{b}^*\|^2 = 1 + (\mathbf{a}^* \cdot \mathbf{b}^*)^2$ , as required for joint measurability. For  $\lambda, \mu \in (0, 1)$ , the square of the norm of the vector  $\mathbf{c}_\lambda$  is

$$\begin{aligned} \|\mathbf{c}_\lambda\|^2 &= \|\mathbf{a}^*\|^2 + \lambda^2 \|\mathbf{b}^*\|^2 + 2\lambda \mathbf{a}^* \cdot \mathbf{b}^* \\ &= \|\mathbf{a}^*\|^2 + \lambda^2 \|\mathbf{b}^*\|^2, \end{aligned} \quad (67)$$

and similarly  $\|\mathbf{d}_\mu\|^2 = \|\mathbf{b}^*\|^2 + \mu^2 \|\mathbf{a}^*\|^2$ , so

$$\begin{aligned} \|\mathbf{c}_\lambda\|^2 + \|\mathbf{d}_\mu\|^2 &= (\|\mathbf{a}^*\|^2 + \|\mathbf{b}^*\|^2) \\ &\quad + \lambda^2 \|\mathbf{b}^*\|^2 + \mu^2 \|\mathbf{a}^*\|^2 \\ &= 1 + \lambda^2 \|\mathbf{b}^*\|^2 + \mu^2 \|\mathbf{a}^*\|^2. \end{aligned} \quad (68)$$

We also have

$$\mathbf{c}_\lambda \cdot \mathbf{d}_\mu = \mu \|\mathbf{a}^*\|^2 + \lambda \|\mathbf{b}^*\|^2. \quad (69)$$

By combining Equations (68) and (69) we see that if  $\mathbf{c}_\lambda$  and  $\mathbf{d}_\mu$  are jointly measurable, i.e.,  $\|\mathbf{c}_\lambda\|^2 + \|\mathbf{d}_\mu\|^2 \leq 1 + (\mathbf{c}_\lambda \cdot \mathbf{d}_\mu)^2$ , then

$$\mu^2 \|\mathbf{a}^*\|^2 + \lambda^2 \|\mathbf{b}^*\|^2 \leq (\mu \|\mathbf{a}^*\|^2 + \lambda \|\mathbf{b}^*\|^2)^2. \quad (70)$$

By expanding, rearranging and noting that  $\|\mathbf{a}^*\|^2 + \|\mathbf{b}^*\|^2 = 1$ , we have that joint measurability of  $\mathbf{c}_\lambda$  and

$\mathbf{d}_\mu$  is equivalent to

$$\begin{aligned} 0 &\leq \mu^2 \|\mathbf{a}^*\|^2 (\|\mathbf{a}^*\|^2 - 1) + \lambda^2 \|\mathbf{b}^*\|^2 (\|\mathbf{b}^*\|^2 - 1) \\ &\quad + 2\lambda\mu \|\mathbf{a}^*\|^2 \|\mathbf{b}^*\|^2 \\ &= -\|\mathbf{a}^*\|^2 \|\mathbf{b}^*\|^2 (\lambda - \mu)^2. \end{aligned} \quad (71)$$

Since we are considering the product of three non-negative quantities, in order for these vectors to be jointly measurable we require  $\lambda = \mu$ . In other words, any pair of vectors  $(\mathbf{c}_\lambda, \mathbf{d}_\lambda)$  of the form given in Eq. (66), with  $\lambda \in [0, 1]$ , correspond to a pair of jointly measurable observables that saturate the bound given in Eq. (59). This categorizes the class of optimal jointly measurable observables with respect to the noise measure when  $\mathbf{a} \perp \mathbf{b}$ . Note that for the metric measure the class of optimal observables are just the vectors of the form  $(\mathbf{c}_0, \mathbf{d}_0)$ , i.e., the smeared target observables, as verified experimentally in [17].

If we now use the probabilistic distance  $D$  to assess Branciard's approximation scheme, we find that it is sub-optimal. Indeed, using Equations (13) and (62), we see that

$$\begin{aligned} D(\mathbf{M}, \mathbf{A}) &= \|\mathbf{a} - \mathbf{m}\| = 2 \sin\left(\frac{\phi - \phi_a}{2}\right), \\ D(\mathbf{M}, \mathbf{B}) &= 2 \sin\left(\frac{\phi_b - \phi}{2}\right) \\ &= 2 \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\phi - \phi_a}{2}\right) \\ &\quad - 2 \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\phi - \phi_a}{2}\right). \end{aligned} \quad (72)$$

At the limit  $\phi = \phi_a$  we see that  $D(\mathbf{M}, \mathbf{A}) = 0$  and  $D(\mathbf{M}, \mathbf{B}) = 2 \sin(\theta/2)$ , and vice versa when  $\phi = \phi_b = \phi_a + \theta$ . By comparison, the values at the limit for the optimal case, as shown in Section VI A are  $(0, \sin \theta)$  and  $(\sin \theta, 0)$ . Since  $\sin \theta \leq 2 \sin(\theta/2)$  for  $\theta \in [0, \pi/2]$ , it follows that the limiting values given by Branciard's scheme are greater than the optimal values for all cases except  $\theta = 0$  (which corresponds to  $\mathbf{A}$  and  $\mathbf{B}$  being already compatible). Indeed, by plotting the curve given by Eq. (72) against the optimal bound given by Eq. (38) (see Fig. 7) we see that, when using the probabilistic distance  $D$  as the measure of error, Branciard's scheme for approximating  $\mathbf{A}$  and  $\mathbf{B}$  is clearly suboptimal.

## B. Ringbauer *et al.*'s experiment

Following on from the work presented by Branciard, Ringbauer *et al.* [16] performed an experimental verification of a measurement scheme which nears the bound (59). To this end they implement a weak measurement protocol as proposed by Lund and Wiseman [19]; this

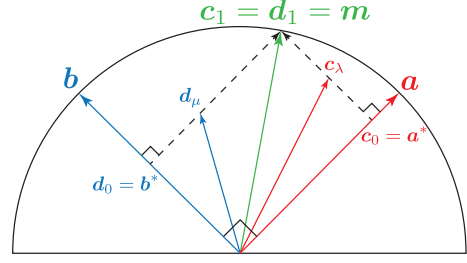


FIG. 6. A pair of lines of observables that saturate the bound given in Eq. (59). In order for  $\mathbf{c}_\lambda$  and  $\mathbf{d}_\mu$  to be jointly measurable we require  $\lambda = \mu$  (which renders  $\mathbf{c}_\lambda - \mathbf{d}_\lambda$  and  $\mathbf{c}_0 - \mathbf{d}_0$  collinear).

method stems from expanding the general form of the noise measure  $\varepsilon(\mathbf{C}, \mathbf{A}, \psi)^2$ :

$$\begin{aligned} \varepsilon(\mathbf{C}, \mathbf{A}, \psi)^2 &= \langle (\mathbf{C}[1] - \mathbf{A}[1])^2 \rangle_\psi + \langle \mathbf{C}[2] - \mathbf{C}[1]^2 \rangle_\psi \\ &= \langle \mathbf{A}[1]^2 + \mathbf{C}[2] \rangle_\psi - 2 \text{Re} \langle \mathbf{C}[1] \mathbf{A}[1] \rangle_\psi. \end{aligned} \quad (73)$$

In our particular case, both  $\mathbf{A}[1]^2$  and  $\mathbf{C}[2]$  are equal to the identity, and so  $\varepsilon(\mathbf{C}, \mathbf{A}, \psi)^2 = 2(1 - \text{Re} \langle \psi | \mathbf{C}[1] \mathbf{A}[1] | \psi \rangle)$ . In most instances it is assumed that the first moment operators  $\mathbf{A}[1]$  and  $\mathbf{C}[1]$  do not commute, and so strictly taking the real part of the expectation value is a necessity here. Following Lund and Wiseman, Ringbauer *et al.* identify the quantity  $\text{Re} \langle \psi | \mathbf{C}[1] \mathbf{A}[1] | \psi \rangle$  as the expectation value of a joint quasi-probability distribution  $p_A^w(k, \ell) = \text{Re} \langle \psi | \mathbf{C}(k) \mathbf{A}(\ell) | \psi \rangle$ . However, whilst this setup was designed to measure both error and disturbance within a sequential measurement scheme—utilizing two probe systems in the process—Ringbauer *et al.* consider a scheme using a single probe and instead simply calculate the noise measure for approximating either  $\mathbf{A}$  or  $\mathbf{B}$ .

In their experimental setup, the incompatible target observables are  $\mathbf{A} = \mathbf{X}$  and  $\mathbf{B} = \mathbf{Z}$ , the spectral measures associated with the Pauli operators  $\sigma_1$  and  $\sigma_3$ , respectively, and the observable being used to approximate them is the sharp observable  $\mathbf{M}$  with associated self-adjoint operator  $M = \cos \phi \sigma_3 + \sin \phi \sigma_1 = \mathbf{m} \cdot \boldsymbol{\sigma}$ . This scenario is equivalent to that presented by Branciard and discussed in the previous subsection, with  $\phi_a = 0$ ,  $\phi_b = \pi/2$ .

Within the context of the paper, a weak measurement of  $\mathbf{A}$  is performed, followed by a measurement of  $\mathbf{M}$ , thereby allowing them to calculate the quasi-probability distribution  $p_A^w$  and hence the value of the noise measure for approximating  $\mathbf{A}$  via  $\mathbf{M}$  (and similarly in the case of  $\mathbf{B}$ ). However, this method is not necessary for our analysis. The weak measurement is solely there to aid in the calculation of the value of the noise measure from the measurement statistics, and does not alter the statistics of the observable  $\mathbf{M}$  that is being used to approximate  $\mathbf{A}$  and  $\mathbf{B}$ .

We recall that the values of the noise measures and

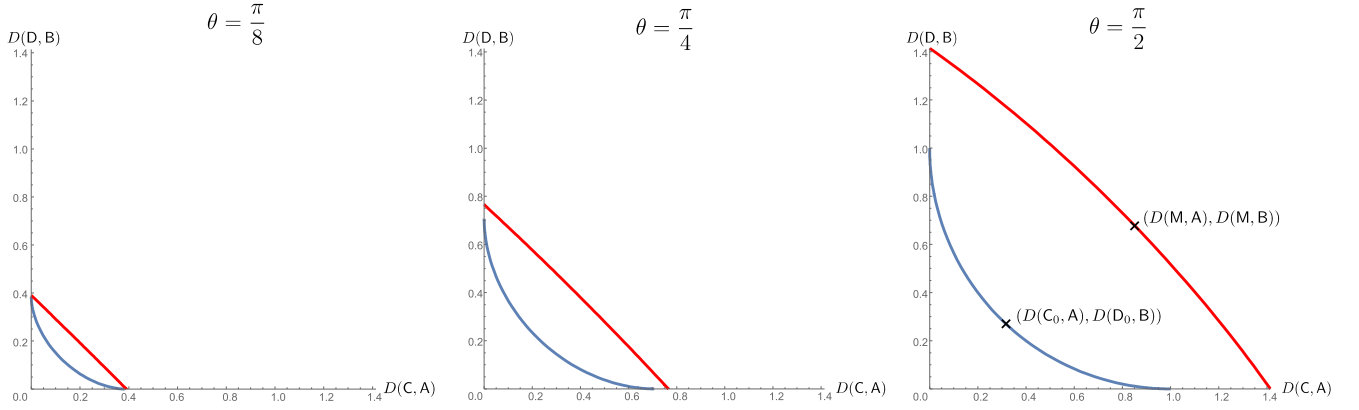


FIG. 7. A comparison of the  $D$  values for the approximating observable  $M$  defined by Branciard's measurement scheme (red curve) against the optimising observables  $C = C_0, D = D_0$ , with values given in Eq. (38) (blue curve). A plot is given for several values of  $\theta$ , corresponding to different degrees of incompatibility between the observables  $A$  and  $B$ . For small  $\theta$  the endpoints of these curves are close to coinciding, but for all other points—corresponding to values of the angle  $\phi$  not equal to  $\phi_a$  or  $\phi_b$ —there is a notable disparity between Branciard's approximators and the optimal case, and indeed for larger values of  $\theta$  even the cases  $\phi = \phi_a, \phi_b$  vary greatly from the optimal situation.

metric errors for  $M$  approximating  $A$  and  $B$  are equal to

$$\begin{aligned} \varepsilon(M, A, \psi)^2 &= 2(1 - \sin \phi) = D(M, A)^2, \\ \varepsilon(M, B, \psi)^2 &= 2(1 - \cos \phi) = D(M, B)^2. \end{aligned} \quad (74)$$

By comparison, the optimal bound for  $D$  (noting that  $\sin \theta = 1$ ) is given by Eq. (44),

$$\begin{aligned} D(C, A) &= 1 - \sin \varphi, \\ D(D, B) &= 1 - \cos \varphi. \end{aligned} \quad (44)$$

It is readily seen that  $D(M, A) \geq D(C, A)$  and  $D(M, B) \geq D(D, B)$ , with  $D(M, A) = D(C, A)$  when  $\phi = \varphi = \pi/2$  and  $D(M, B) = D(D, B)$  when  $\phi = \varphi = 0$  (see the rightmost plot in Fig. 7).

### C. Optimal approximators for metric error

The optimal approximators given by Yu and Oh arise naturally within the derivation of their optimal bound. Starting from Equations (34) we rearrange in order to acquire  $\mathbf{c}$  and  $\mathbf{d}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$ , ultimately arriving at

$$\begin{aligned} \mathbf{c} &= \frac{(D(D, B) + (1 - M^2) \cos \varphi) \sin \varphi \mathbf{a} + MD(C, A) \cos \varphi \mathbf{b}}{\sin \theta}, \\ \mathbf{d} &= \frac{(D(C, A) + (1 - M^2) \sin \varphi) \cos \varphi \mathbf{b} + MD(D, B) \sin \varphi \mathbf{a}}{\sin \theta}, \end{aligned} \quad (75)$$

where  $M$  is given by Eq. (37). These vectors define the (necessarily) jointly measurable dichotomic observables  $C$  and  $D$  via

$$C(\pm) = \frac{1}{2}(I \pm \mathbf{c} \cdot \boldsymbol{\sigma}), \quad D(\pm) = \frac{1}{2}(I \pm \mathbf{d} \cdot \boldsymbol{\sigma}), \quad (76)$$

whose joint observable  $J$  is of the form

$$J(k, \ell) = \frac{1}{4}(1 + k\ell M)I + \frac{1}{4}(k\mathbf{c} + \ell\mathbf{d}) \cdot \boldsymbol{\sigma}, \quad (77)$$

with  $k, \ell = \pm 1$ .

Due to their reliance on the angle  $\varphi$  it is not very clear what form the vectors  $\mathbf{c}$  and  $\mathbf{d}$  take, but it is instructive to see that in the case  $\theta = \pi/2$  (where  $A$  and  $B$  are maximally incompatible),  $\mathbf{c}$  and  $\mathbf{d}$  reduce to

$$\begin{aligned} \mathbf{c} &= (D(D, B) + \cos \varphi) \sin \varphi \mathbf{a} = \sin \varphi \mathbf{a}, \\ \mathbf{d} &= (D(C, A) + \sin \varphi) \cos \varphi \mathbf{b} = \cos \varphi \mathbf{b}, \end{aligned} \quad (78)$$

where we have made use of Eq. (44) and the fact that  $M = 0$  according to Eq. (37). In other words, the observables  $C$  and  $D$  are smeared versions of the sharp observables  $A$  and  $B$ , respectively, which coincides with what we would expect for maximally incompatible observables when we use the metric error measure [3]. Furthermore, we see that  $\varepsilon_A^2 = 2(1 - \mathbf{a} \cdot \mathbf{c}) = 2(1 - \sin \varphi)$  and  $\varepsilon_B^2 = 2(1 - \cos \varphi)$  for  $\theta = \pi/2$  (confirming Eq. (65) for  $\lambda = 1$ ), and so by making use of these values in the left hand side of Eq. (59), where now  $\sin^2 \theta = 1$ , we see immediately that this case optimizes the bound given by Branciard.

However, these approximators only saturate the bound in the case  $\theta = \pi/2$ . If we plot the left-hand side of Eq. (59), using the Bloch vectors in Eq. (75), against the right-hand side for different values of  $\theta$ —see Fig. 8—then we see that these observables are suboptimal with the exception of  $\theta = \pi/2$ .

## VIII. CONCLUDING DISCUSSION

This paper has provided a thorough examination of two quantities commonly used as measures of error in quantum mechanics, namely the metric error  $D$  and the noise measure  $\varepsilon$ ; the aim of this was to find the optimal error bounds for both measures when approximating two incompatible dichotomic target observables  $A, B$ , on

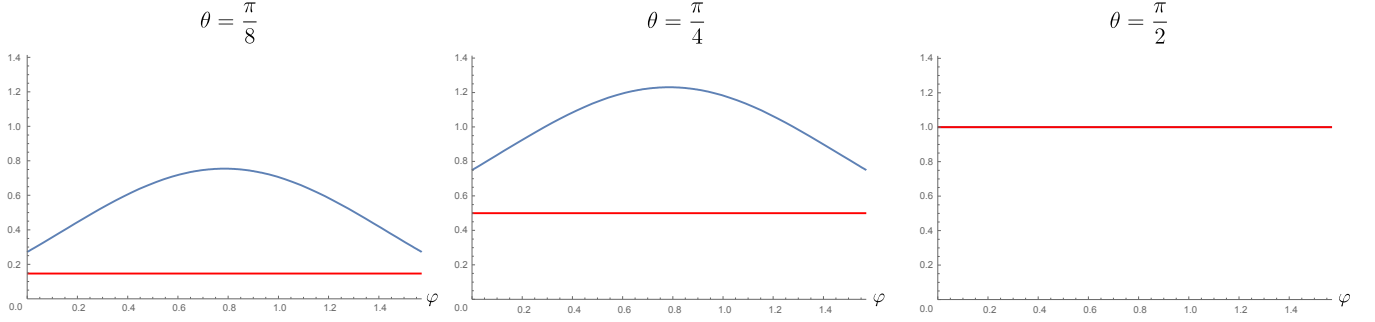


FIG. 8. A plot of the left-hand side of Branciard’s inequality (59) in the case of Yu and Oh’s optimal approximating observables for various values of  $\theta \in [0, \pi/2]$ , shown by a blue curve, compared against the lower bound ( $\sin^2 \theta$ ), which is the red line. With the exception of the maximally incompatible case,  $\theta = \pi/2$ , the blue curve is always a greater value than the red line, highlighting that Yu and Oh’s optimal approximators are suboptimal in terms of the noise measure.

a qubit system via jointly measurable dichotomic observables  $C$  and  $D$ .

The disparity between the choice of noise measure and metric error reflects a fundamental difference in the understanding of the notion of an “observable”, with the proponents of the noise measure attributing observables to self-adjoint operators (and by extension their unique PVMs), while the users of the metric error rely on the more general construction of POVMs. Depending on these different readings of “observables”, the subsequent task of finding jointly measurable approximating observables takes on different forms, leading to the version given by Hall in the case of the noise measure, and the one based on the construction of joint probabilities presented in Sec. II. However, whilst these constructions are conceptually different—the former being a post-processing task and the latter built around finding an observable for which the approximating observables appear as margins—we find that they are mathematically equivalent if cast in terms of POVMs, thereby allowing for the subsequent comparison of the two error measures.

The minimum error curves for approximating  $A$  and  $B$  by  $C$  and  $D$ , respectively, were found by solving constrained optimization problems. In the case of the metric error, this was previously done by Yu and Oh, but we have provided some technical clarification, including a physical interpretation, to the minimum error curve.

When attempting this optimization problem for the noise measure, by using the method of Lagrange multipliers, we find that a local solution is only possible in the case where the target observables  $A, B$  are maximally incompatible. By applying geometric arguments, we see that the optimal observables for the noise measure are the extremal (sharp) observables whose Bloch vectors lie on the surface of the circle spanned by the Bloch vectors of  $A$  and  $B$ , regardless of the degree of incompatibility between the target observables.

However, in the case of maximally incompatible target observables ( $\theta = \pi/2$ ), we find a continuous family ( $C_\lambda, D_\lambda$ ) of pairs of optimal observables (the extremal

ones included as  $M = C_1 = D_1$ ) (see Fig. 6). In this case the (unique) pair of optimal approximating observables found for the metric error is also optimal with respect to the noise measure (they are given as the pair  $(C_0, D_0)$ ).

There is a fundamental difference at this point: The errors  $D(C_\lambda, A)$ ,  $D(D_\lambda, B)$  increase monotonically with  $\lambda \in [0, 1]$ , reflecting the increasing dissimilarity of the observables  $C_\lambda, A$  and  $D_\lambda, B$ , seen in their increasingly divergent statistics. In contrast the noise values  $\varepsilon(C_\lambda, A, \psi)$ ,  $\varepsilon(D_\lambda, B, \psi)$  are constant. This constitutes the challenge for the proponents of the noise quantity  $\varepsilon$  as an error measure to identify an operational criterion which explains in which sense one could consider all members of (say) the family  $C_\lambda$  as equally good approximators of  $A$ . This constancy is related to the fact that the noise measure double-counts the unsharpness of the approximating observables, as previously noted in [3] and apparent here by considering Eq. (14) and Fig. 6.

We revisited another deficiency of the noise measure: the existence of approximating observables  $C$  of the target observable  $A$  that do not commute with  $A$  and yet lead to  $\varepsilon = 0$ . This was found via the post-processing method of Hall that is applied to minimize  $\varepsilon$ . It seems to us that such post-processing does nothing to improve the accuracy of the measurement being performed; instead, our example shows that it may lead to a false indication of a (non-existent) perfect measurement accuracy.

We also observed that one can get  $\varepsilon(C, A, \psi) = 0$  for approximators  $C$  that commute with the target observable. This was shown to be possible by constructing a family of biased observables, and we saw that  $\varepsilon = 0$  could only arise when  $\psi$  is an eigenstates of  $A$ . Then, as an extreme case, one could simply set the approximating observable to be a trivial observable that amounts to doing nothing other than declaring the known outcome—an approximation scheme of little interest or use. Ozawa’s proposed remedy for this “false positives” deficiency of  $\varepsilon$ —the introduction of his local uniform measure  $\bar{\varepsilon}$ —was seen to display undesirable discontinuity features in the present context of qubit measurements.

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